STRESS & DEFLECTION IN AN ELASTIC MASS UNDER SEMIELLIPSOIDAL LOADS

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by

J.L. SANBORN

Joint Highway Research Project

PURDUE UNIVERSITY
LAFAYETTE INDIANA

Informational Report

STRESS AND DEFLECTION IN AN ELASTIC MASS
UNDER SEMIELLIPSOIDAL LOADS

TO: K. B. Woods, Director
Joint Highway Research Project

May 8, 1963

FROM: H. L. Michael, Associate Director
Joint Highway Research Project

File: 6-20-6
Project: C-36-52F

Attached is an informational report entitled "Stress and Deflection in an Elastic Mass Under Semiellipsoidal Loads" which has been prepared by Mr. John L. Sanborn a member of the staff of the School of Civil Engineering. The research reported was the thesis of Mr. Sanborn for the MSCE degree and was performed under the direction of Professor E. J. Yoder.

The research reported is concerned with the estimation of stresses and deflections at points within a pavement due to applied wheel loads. Specifically the report describes the application of numerical integration, by means of a high speed digital computer, to the solution of elastic stress and deflection equations for a semiellipsoidal load at the surface of a uniform, semi-infinite mass having a plane boundary.

The research was financed by the School of Civil Engineering and the University and is presented to the Board for information.

Respectfully submitted,

Harold L. Michael

Harold L. Michael, Secretary

HLM:kmc

Attachment

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F. F. Havey
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M. B. Scott
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J. L. Waling
E. J. Yoder

Informational Report

STRESS AND DEFLECTION IN AN ELASTIC MASS UNDER SEMIELLIPSOIDAL LOADS

by

John L. Sanborn
Research Assistant

Joint Highway Research Project
File No: 6-20-6
Project No: C-36-52F

Purdue University
Lafayette, Indiana

May 8, 1963

ACKNOWLEDGMENTS

The author expresses his deep appreciation to the many people who gave support, encouragement and advice throughout the conduct of this study.

Especial thanks is extended to Professor E. J. Yoder who served as the author's major professor. Professor Yoder provided the original suggestion for the study and contributed much through his continuous interest and encouragement, as well as his reviews of the several stages of the work.

Thanks is expressed also to Professor T. J. Herrick and the School of Aeronautical and Engineering Sciences for guidance and for use of a computer during initial studies, and to the Computer Sciences Center for funds, services and equipment to carry out the project.

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ABSTRACT

Sanborn, John Leonard, MSCE, Purdue University, January 1963. Stress and Deflection in an Elastic Mass Under Semiellipsoidal Loads. Major Professor: Eldon J. Yoder.

A common problem in the design and evaluation of highway and airfield pavements is the estimation of theoretical stresses and deflections at points within the pavement due to applied wheel loads. Elastic stress distribution in a homogeneous mass is generally used for such estimates, but its application in this field has been limited to assumptions of concentrated load or of uniformly distributed circular loads. Formal integration of the expressions for stress and deflection for non-uniform loads is either impossible of impracticable.

This report describes the application of numerical integration, by means of a high speed digital computer, to the solution of elastic stress and deflection equations for a semiellipsoidal load at the surface of a uniform, semi-infinite mass having a plane boundary. Normal stresses are computed by the Boussinesq equations, and the strain due to those stresses determined by accepted elastic theory. Integration is performed by Simpson's Rule. Three sets of curves representing stress and deflection on three planes normal to the surface are included. The curves were compiled from data developed, with the program described, on an IBM 7090 computer.

The actual program, in 7090 Fortran language, and an example of the use of the curves are appended to this report.

INTRODUCTION

One of the most frequently encountered problems in design and evaluation of flexible highway and airfield pavements is the estimation of stresses and deflections at points within the structure and subgrade due to applied wheel loads. Recent studies (19, 21)* have shown, in particular, the importance of determining deflections of various components of pavements in order to evaluate their adequacy or serviceability. In the past, a common approach to this problem has been to consider the pavement and subgrade to act as an elastic mass and to evaluate stresses according to the Boussinesq theory of stress distribution. This approach implies, of course, a significant simplification of the layered systems actually used in pavements. The numerical values of stress and deflection determined from this simplification vary significantly from true values both as a result of the idealization of the structure and of the difficulty of determining elastic properties of the materials. Nevertheless, patterns of measured stress and deflection do follow theoretical values (21). The idealized case may, therefore, be used reasonably to evaluate relative effects of different loads or different pavements. Basic equations for stress due to a concentrated load are integrated over the loaded area to approximate the effect of a wheel load. Strains integrated from any given depth to infinity will give total elastic deflection of the given point.

The major difficulty with such a procedure is that it has been

* Numbers in parentheses refer to references listed in Bibliography.

practically limited to uniform circular loads because of the unmanageable expressions which it is otherwise required to integrate. A partial solution is provided by influence charts (12, 13) prepared by Newmark based on the effects of uniform circular loads. These permit determination of stresses and deflections at any point due to any uniform load. Even this requires different scale drawings for each load at each depth for which stress and/or deflection is required, however. Furthermore, non-uniform loads require superposition of a series of uniform loads to approximate the effect of the actual load.

Curves of stress and deflection have also been produced (4), based on Newmark's influence charts, which are practical to use from a design or investigation standpoint. These, however, serve only the uniform circular load.

It has long been known that wheel loads vary considerably from circular areas of uniform pressure, (7), but no feasible method of evaluating the effect of these loads has been available. With the advent of high speed digital computers, however, numerical methods of integrating any configuration of load are entirely practicable. An important use of this tool is to compare the effects of known load distributions to those of conventionally assumed distributions, and to provide simple and useful techniques for estimating those effects in practice. To this end, the computer program reported here was devised to evaluate vertical stress and deflection according to Boussinesq theory due to semiellipsoidal loads. The completed program was used to prepare curves of stress and deflection which are included herein.

PURPOSE AND SCOPE

The purpose of this study was to devise a numerical solution for theoretical stresses and deflections in a soil mass due to non-circular loads of non-uniform distribution. The analysis was based on accepted theory of elasticity and previous investigations of the configuration of ordinary wheel loads.

The study was composed of two basic parts--(1) to write a program for a digital computer, which could be used directly or with minor modifications for any given load, and (2) to use that program to develop charts of vertical stress and deflection for a reasonable approximation of usual tire loads.

The computer program has been written in the IBM Fortran language which is acceptable to a variety of digital computers readily available to many highway and airfield agencies as well as to most universities and colleges. Variations are easily introduced to provide for different load conditions.

Charts of stress and deflection were developed for a semiellipsoidal load of proportions which approximate contact pressures and areas under some standard tires. The charts are in the form of curves of stress and deflection, respectively, in terms of maximum contact pressure, plotted against depth in terms of size of contact area. Deflection is treated as a factor, since actual deflections depend on the contact area and the elastic modulus of the soil as well as the contact pressure.

Stresses are considered to be distributed according to Boussinesq

theory which assumes a homogeneous, isotropic elastic mass. No attempt is made to account for the effects of the layered structure of flexible pavements. Several investigators have described solutions to two and three layer problems. However, since a primary function of this program is to provide a basis for comparison of effects of non-uniform loads with the conventionally assumed uniform circular loads, the work was restricted to the simpler, homogeneous case which has been the assumption adopted by other investigators in a large portion of the earlier work.

REVIEW OF LITERATURE

A number of investigations have shown that the assumption of elastic behavior is reasonable for the response of flexible pavements to traffic loads. Foster and Fergus (5) compared results of extensive test measurements on a clayey silt subgrade to theoretical stresses and deflections and reported good agreement. Newmark's influence charts (12, 13) have been used extensively, and served as the basis for Foster and Ahlvin's charts (4) of stress and deflection due to a circular load of uniform pressure distribution. The latter charts are commonly used in design and research problems in highway and airfield pavements (20). Palmer and Barber (14) also made use of the assumption of elastic behavior with satisfactory results. Baker and Papazian (1), Walker (19) and Yoder (21) all point out the important contribution of the elastic portion of pavement deflections in determining the performance of pavement.

Burmister (3) has considered the effect of the layered structure of pavements and presented curves of surface deflection at the center of a uniform circular load for the two layer case. Fox (6) extended the work of Burmister to evaluate stresses at various points in the structure. More recently, with the use of a digital computer, Jones (9) has developed tables of stresses for a range of conditions of a three layer structure. These are all based on a uniform circular load at the pavement surface, and are limited to points on the axis of symmetry of the load.

There have been relatively few studies of tire contact areas and pressures. However, results of several studies indicate significant variations from the assumption of a uniform circular load. Among the early investigators were Heldt (7), Teller and Buchanan (17) and McLeod (11), all of whom note the nearly elliptical contact area of truck tires. More recently, Brahma (2) and Lawton (10) have made the same observations with respect to truck and aircraft tires respectively.

Lawton's work is of particular interest in that both contact areas and pressures were determined for loads on a simulated subgrade rather than on a rigid surface. The subgrade used was a mechanical model which has been described in detail by Herner and Aldous (8). The contact areas for a variety of tires and loads were all elliptical and had a reasonably uniform ratio of major to minor axis, averaging about 1.7. Pressure distribution was ellipsoidal for all loads within rated capacities of the tires, and varied to more nearly uniform at overloads.

With respect to numerical methods for calculating stresses and deflections in an elastic mass, Stoll (15) has used a digital computer for calculation of vertical stresses in a foundation soil due to variable and asymmetric column loads. His results indicated that the approach is satisfactory and feasible for use in similar problems.

PROCEDURE

Computer Program

Program Logic

The procedure used in this study was to evaluate stresses in an elastic mass by numerical integration of the Boussinesq equations, and to compute elastic strains resulting from those stresses. The equations attributed to Boussinesq express stress components at any point within a semi-infinite, elastic, isotropic, homogeneous mass due to a single concentrated load on the plane boundary of the mass. With reference to Figure 1, the stress components, as given by Taylor (16), are

$$\sigma_z = \frac{P}{2\pi}\left[\frac{3z^3}{(r^2+z^2)^{5/2}}\right] \tag{1}$$

$$\sigma_r = \frac{P}{2\pi}\left[\frac{3r^2z}{(r^2+z^2)^{5/2}} - \frac{(1-2\mu)}{r^2+z^2+z\sqrt{r^2+z^2}}\right] \tag{2}$$

$$\sigma_t = -\frac{P}{2\pi}(1-2\mu)\left[\frac{z}{(r^2+z^2)^{3/2}} - \frac{1}{r^2+z^2+z\sqrt{r^2+z^2}}\right] \tag{3}$$

$$\tau_{rz} = \frac{P}{2\pi}\left[\frac{3rz^2}{(r^2+z^2)^{5/2}}\right] \tag{4}$$

where μ is Poisson's ratio for the elastic mass. A derivation of these equations is presented by Timoshenko and Goodier (18).

The elastic strain of a point subjected to stress is determined from the normal components of stress by the equation

$$\delta_1 = \frac{1}{E}\left[\sigma_1 - \mu\sigma_2 - \mu\sigma_3\right] \tag{5}$$

where μ is Poisson's ratio and E is the modulus of elasticity of the

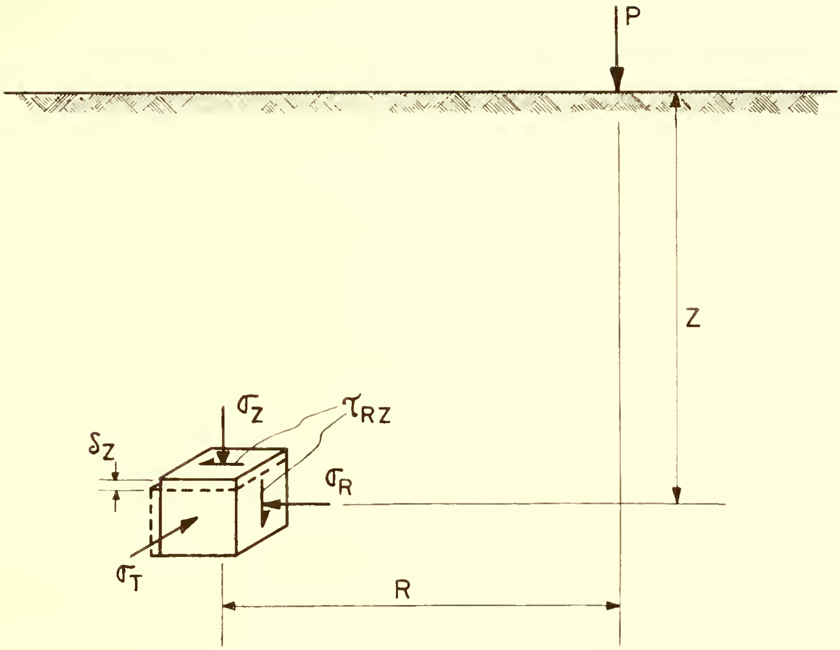

FIGURE I. STRESSES ON AN ELEMENT DUE TO A CONCENTRATED LOAD AT THE SURFACE

material. Total deflection of a given point depends not only on the strain of that point, but also on strains of other points deeper in the mass. Therefore, to get useful deflections it is necessary to integrate the strains on a vertical line from the given point to infinity. In a numerical procedure the summation is carried out to such depth that the remainder is negligible.

In applying these equations to distributed loads, the basic process is to express the stresses and strain due to an infinitesimal element of the load, treated as a single concentrated load, and to integrate those expressions over the entire loaded area. It is evident that except for symmetrical loads of uniform distribution the formal integration of the expressions is extremely awkward and impractical, even if it is possible, for a given load configuration. A numerical process, however, using a high speed digital computer, is entirely feasible for any load configuration, the only requirement being to define the area of load and the distribution of pressure over that area. In this process, the load is divided into a number of finite elements, each element being treated as a concentrated load, and the numerical integration is carried out in two coordinate directions over the entire loaded area.

The numerical integration used in this program is Simpson's Rule -

$$S = \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots y_{n-2}) + y_n\right] \quad (6)$$

where y_i are values of the function at equally spaced intervals, h is the spacing and n is the number of spaces (must be even).

Figure 2 illustrates the terminology and arrangement used in development of this program. Plane XY is the boundary of the elastic mass under consideration. Surface ABCDE represents the distributed load

applied to the boundary, and Q, a point in the mass at which it is required to determine stress and deflection. Coordinates G and H locate the vertical line through Q with respect to the coordinate axes XYZ, and point X_{IJ}, Y_{IJ} is any element of the load. P_{MAX} represents the maximum ordinate of the pressure distribution and P_{IJ} the pressure at point X_{IJ}, Y_{IJ}. R_{IJ} is the horizontal distance from the vertical line through Q to point X_{IJ}, Y_{IJ}, and is defined by the equation

$$R_{IJ} = \sqrt{(X_{IJ} - H)^2 + (Y_{IJ} - G)^2}. \tag{7}$$

The program as described here provides for the surface ABCDE to be semiellipsoidal, although the program could be modified to permit any desired load distribution. Vertical stress is determined as a fraction of P_{MAX}, the reference pressure. Deflection is treated as a factor F in the equation

$$\Delta = \frac{P_{MAX}B}{E} F \tag{8}$$

where B is the reference dimension and E is the modulus of elasticity of the soil. Flow of the entire program is shown in block diagram form in Figure 3. A complete Fortran program for the IBM 7090 is given in Appendix A.

The first step in the program is to map the pressure distribution, P_{IJ}, and compute the radial distances R_{IJ} for each element of load, and store these quantities in two arrays in computer memory. It should be noted that for Simpson's Rule to apply an even number of equal intervals, and therefore an odd number of points in the loaded area, is necessary. Computation of stresses and deflection then proceeds along vertical line QS (Figure 2), beginning at the greatest depth used and working upward. This order of computation is used in order to

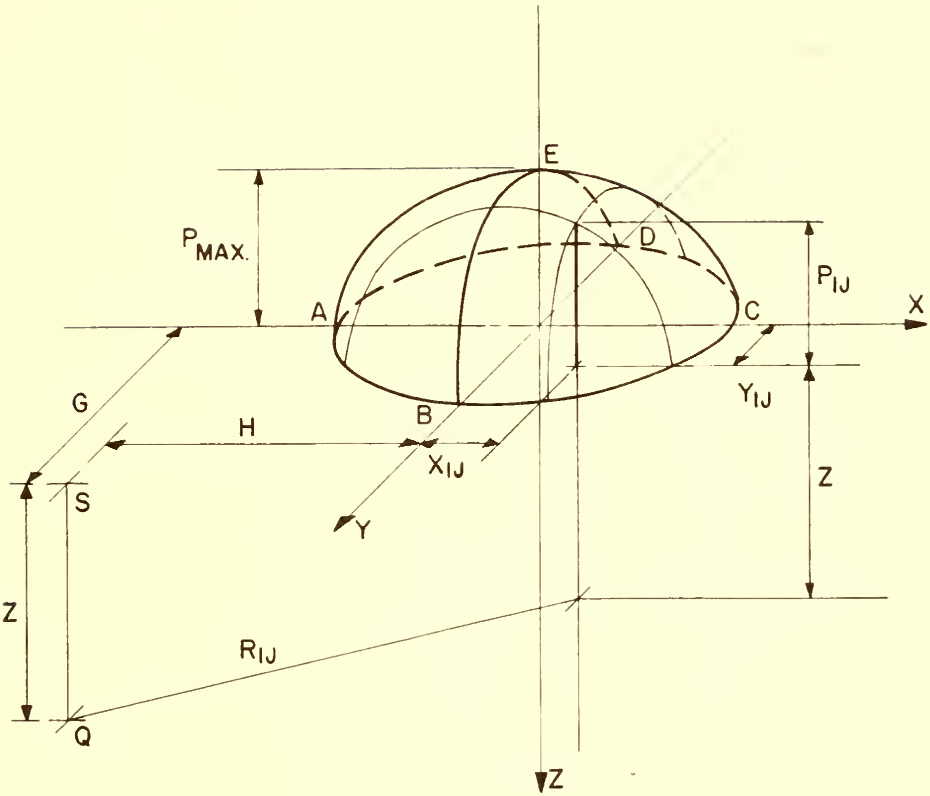

FIGURE 2. A DISTRIBUTED LOAD ILLUSTRATING TERMINOLOGY USED IN FORTRAN PROGRAM

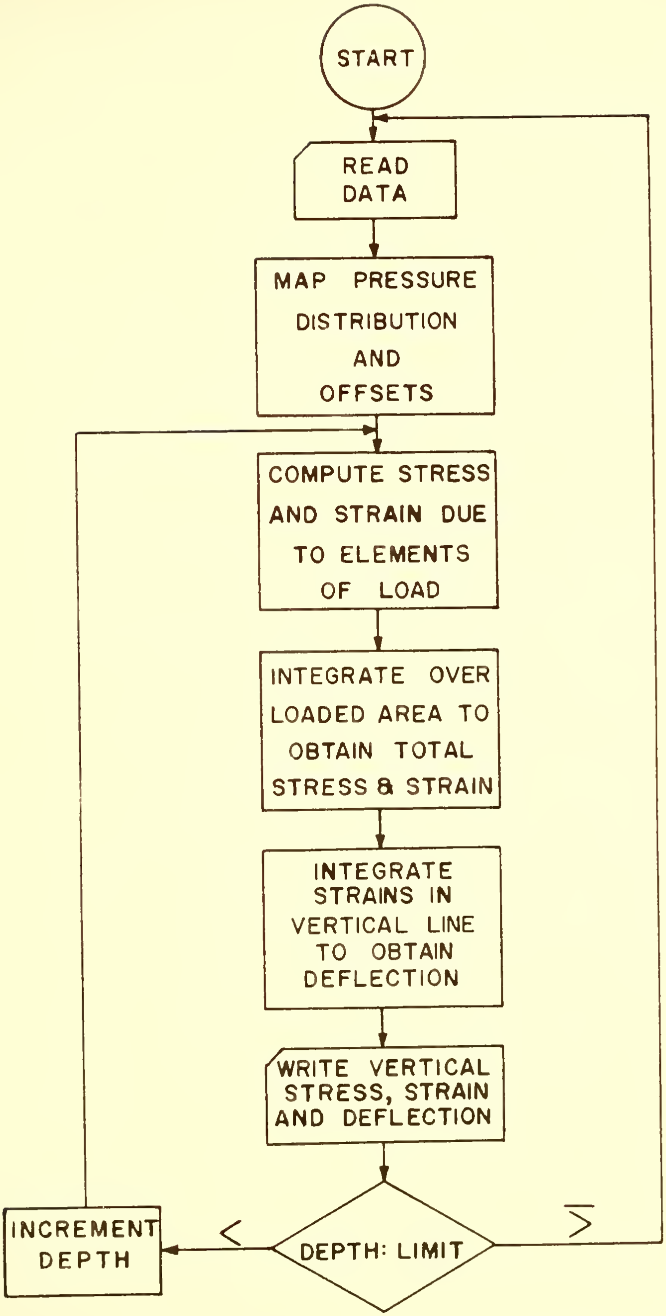

FIGURE 3. FLOW DIAGRAM

facilitate integration of strains and to eliminate the requirement of storing results in memory until a vertical line is completed.

Because the orientation of the point relative to an element of the load is variable, it is necessary to compute strain directly due to each element of the load. The program, therefore, provides for computation of both stresses and strain at a single point of interest due to a single element of load, repeating this routine for each element of load, then sums the results for the given point of interest. Similar computations for successive points along line QS permit the determination of total deflections.

Calculation of stresses for each element of load is according to equations 1, 2, and 3. Shear stress was not included since it has no effect on vertical strain, though it could be included easily provided a computer with adequate memory is available. With normal stresses known it is possible to calculate the factor $\sigma_1 - \mu\sigma_2 - \mu\sigma_3$ in equation 5. This permits the factor 1/E to be applied independently for the soil conditions pertaining in any instance. Because the strain required is that in a vertical direction, the factor becomes

$$\sigma_z - \mu\sigma_r - \mu\sigma_t$$

Having determined stress and the strain factor at Q due to one element of load, these values are stored in computer memory and the routine is repeated for the next element in the array. When all elements in one line of one coordinate direction have been covered, the vertical stress and the strain factor are integrated by equation 6 and the result stored in memory. The same process is used for successive lines across the loaded area. When all lines of load elements have been covered, the

stress and strain factor of the total load is calculated by another application of equation 6 in the second coordinate direction. The total stress and strain factor at the given point are then in computer storage available for printing out.*

Total deflection is, as noted, the sum of all strains along the vertical line through the point in question. Because Simpson's Rule requires an even number of intervals, it is not possible to perform the integration at every point. The computation of deflection factor is, therefore, limited to alternate points on the vertical line. Assuming zero deflection below the point of beginning, an initial deflection is calculated for the first point. Depth, stress, strain and deflection factor are printed out for this point. Depth is then reduced by a predetermined increment and the program returns to the point of computing normal stresses. When stress and strain for this point are computed, they are printed out, with no deflection computation, and depth is incremented again. On completing the third run, deflection is again computed, taking into account strains of all points computed previously, and again, depth, stress, strain and deflection factor are printed out. This process is repeated continuously, computing deflection factor only at alternate points, until depth is essentially zero. No actual result can be calculated for depth equal to zero at points beneath the load, because stress expressions for this case become undefined (i.e., zero divided by zero). End points for the stress curves are known, however, from the load distribution.

* Horizontal components of stress were not stored for summation because of lack of memory in the IBM 1620, on which the original programming was done. It should be noted, however, that these stresses could be retained and printed out with larger machines such as the IBM 7090 by adding subscripts to their notation, reducing to a fixed orientation, and repeating the summation process for these items.

This completes one cycle of program operation. Program control then causes reading of a new data record which defines a new vertical line and may provide new load configuration, grid size and/or Poisson's ratio. The computer is automatically reset to initial conditions and the entire program is repeated. Operation continues until no further data are supplied.

Provision is made in the program for a series of three different depth increments. This is desirable because the curves have relatively sharp curvature near the surface but are nearly straight at considerable depth. Using large increments for the deep points, a minimum of computer runs, and therfore computer time, is required. In areas of sharper curvature, the smaller intervals necessary for proper definition of the curves are used.

Input

Two input formats are required by this program. One provides data relative to the load and the vertical line along which stresses and deflection factors are determined. The second provides the depth bounderies which define the area of computation and the depth increments in each interval.

Data in the first input record are: A, the ratio of major to minor axis of the contact area (this may be unnecessary, or inadequate, for other distributions of load); N and M, the grid size; H and G, the coordinates which locate the vertical line along which computation proceeds; μ, the value of Poisson's ration to be used in the computations; and IDENT, an identification number to control flow of the program. The second input record includes DZ1, DZ2, DZ3, ZF1, ZF2, and ZF3 which

are the increments of depth and limiting depths, respectively, for each of three intervals. Computations begin at ZF3, and depth is incremented by DZ3. When depth ZF2 is reached, the increment is reduced to DZ2, and at depth ZF1 the increment is reduced again to DZ1. The use of two formats permits keeping input to a minimum. Having once established the depth ranges and increments, further data can be processed without redefining these values. On the initial data card a negative digit is punched in the control word IDENT. This digit causes reading of the following card which must contain the depth data in the second format described. Operation then proceeds normally. In successive data records, as long as the depth limits are to remain unchanged, the identification word is left blank. Program operation then by-passes the second input block, proceeding directly with computations for the new line of interest. New depth controls can be introduced for any set of data simply by punching a negative digit in the identification word of that data card and following it with the appropriate depth control data in the specified format.

All linear measurements are in terms of a characteristic dimension of the loaded area. This provides dimensionless results for compilation of general figures. For specific locations or deflections, however, results must be multiplied by the appropriate dimension.

Output

As each set of input data is read, that set of data is printed out for identification of the results which follow it. Each successive computer run through the program results in one line of printed output which represents conditions at a single point of interest. Information

printed includes depth of point as well as vertical stress and vertical strain at that point. Alternate lines also include the total deflection factor for the point. Thus, complete operation on one data set provides in printed output the coordinates of a vertical line relative to the loaded area, and a series of results for points along that line.

When all points defined on a vertical line have been completed, the program automatically reads a new data card representing a new line of interest. The process is repeated continuously as long as data are supplied to the computer.

The vertical stress indicated in the output represents the actual normal component of stress, at the point described, as a fraction of the maximum contact pressure, P_{MAX}. Deflections are expressed as a factor, F, which must be applied in equation 8 to give actual total deflections.

Curves

The second phase of this study, following development of the computer program, was to develop a set of curves of vertical stress and of deflection factor that would be useful both for design application and for further study. A number of difficulties in this aim are immediately evident, among the most important of which are -

1. Every distribution of load will produce a different family of curves;
2. Asymmetric loads will produce a unique family of curves for each vertical plane through any reference point;
3. Curves of deflection will depend on the value of Poisson's ratio used; and

4. Some characteristic pressure must be known as a base for determining actual stress and deflection for any given load.

There is nearly general agreement among investigators that the contact area under wheels on flexible pavements is approximately elliptical. Lawton's work on aircraft tires (10) confirms this and also shows that, for loads within the rated capacity of a tire, the distribution of pressure is essentially semiellipsoidal. The proportions of the contact area are fairly constant, the ratio of major to minor axis ranging generally from 1.6 to 1.8. Maximum pressure depends both on tire pressure and total load.

Typical curves of maximum pressure, P_{MAX}, vs. total load are shown in Figure 4. It can be seen from these curves that at rated load, on most subgrades, P_{MAX} is practically constant at 132 per cent of the average rigid plate contact pressure. In general, the average pressure may be taken as the inflation pressure. Thus, P_{MAX} would be 1.32 times the inflation pressure. For more precision, of course, it may be necessary at times to refer to actual curves of contact pressure vs. inflation pressure and P_{MAX}/p_{ave} vs. load, as presented by Lawton.

On the basis of available information it was decided to construct a set of curves for a semiellipsoidal load, using a ratio of major to minor axis of contact area equal to 1.7. This is shown in Figure 5 where A = 1.7 and B = 1. The equation of the surface which defines the pressure distribution is

$$\frac{X^2}{A^2} + \frac{Y^2}{B^2} + \frac{P^2}{P_{MAX}^2} = 1 \qquad (9)$$

Hence, the pressure at any point, (I, J), on the contact area is:

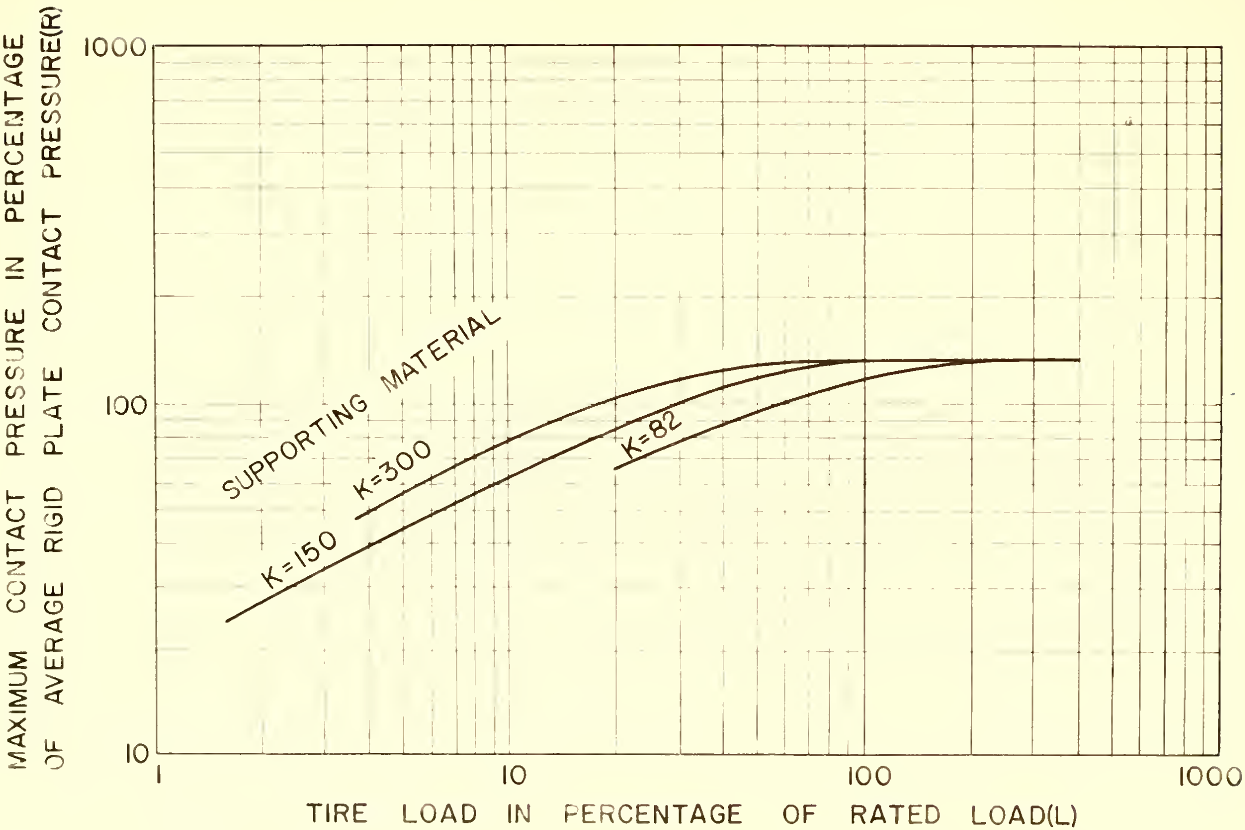

FIGURE 4. MAXIMUM CONTACT PRESSURE AT CENTER OF TIRE
(FROM LAWTON, PROCEEDINGS, HIGHWAY RESEARCH BOARD, 1957)

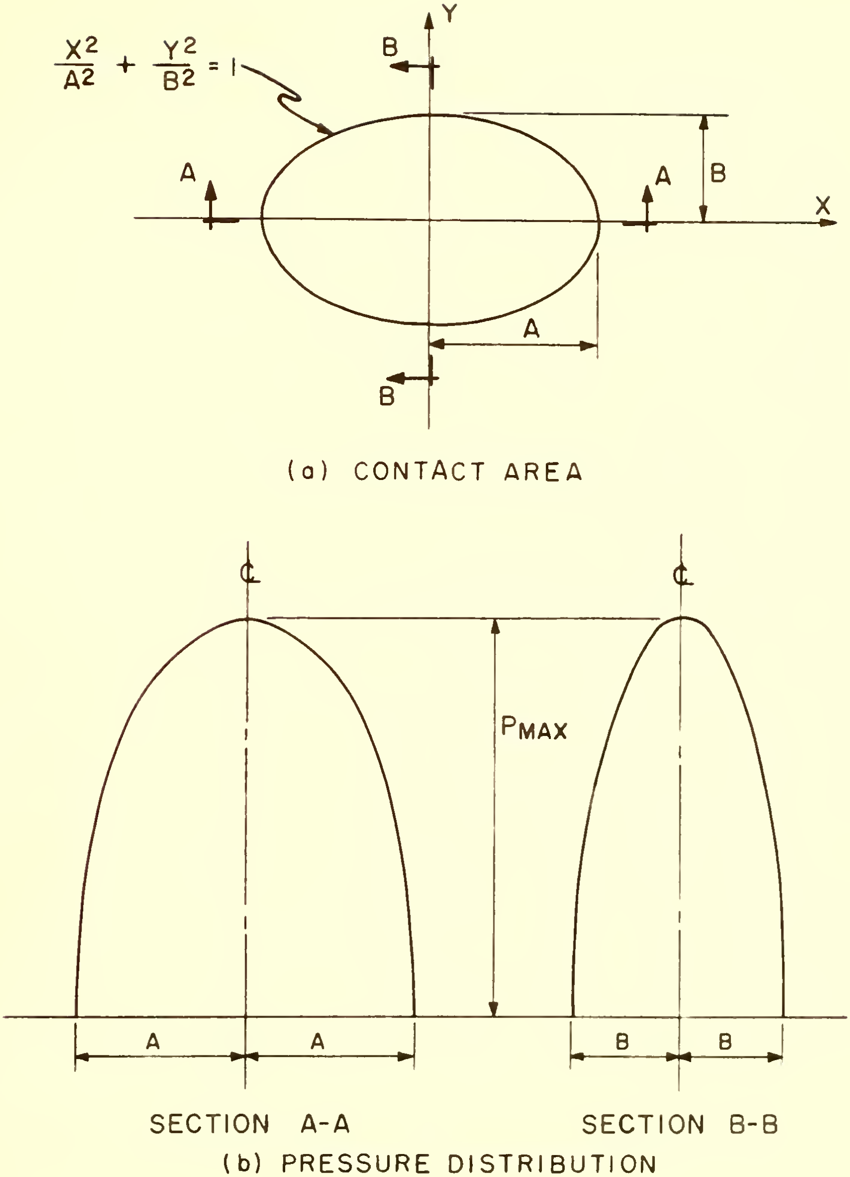

FIGURE 5. CONTACT AREA AND PRESSURE DISTRIBUTION

$$P_{IJ} = P_{MAX}\sqrt{1 - X_{IJ}^2 - \frac{Y_{IJ}^2}{A^2}} \tag{10}$$

No attempt was made in this study to evaluate Poisson's ratio for any particular condition. While the computer program provides for any value between 0 and 0.5, the development of the curves was in conformance with the general practice of taking $\mu = 0.5$. The curves for vertical stress are not dependent on μ, of course, so are valid for any soil. The curves for deflection factor are dependent on μ as is seen in equations 2, 3, and 5, and are, therefore, limited to the assumption of $\mu = 0.5$ which implies no volume change.

The question of orientation is not as easily resolved. Because the load is not distributed symmetrically, the stress, and therefore deflection, a given distance from the center in one direction is different from that in another direction. For single or dual wheels, one set of curves would be adequate since all points of interest would be under the center of the load or on a plane through the transverse axis. When considering tandem axles, however, two more planes are immediately necessary - one along the longitudinal axis of one wheel and one through the centers of two diagonal wheels. Theoretically, this would require a different diagonal plane for each geometrically different arrangement of tandem axles in use. It would be possible to develop a complete set of charts for planes oriented say every five or ten degrees from the transverse axis. However, since one wheel of a tandem gear has a relatively small effect on stresses under the diagonally opposite wheel, the result would seem to be unwieldly and confusing, without adding any great usefulness to the charts. There-

fore, one diagonal plane was selected which would give a reasonable approximation of the diagonal for standard dual-tandem gears. It is expected that for most practical situations the actual stresses and deflections determined from this family of curves will give values correct within the precision of the curves. When more precise work is needed, of course, the computer program may be used directly or a new set of curves may be produced for the required orientation.

The four major problems are thus resolved by assuming a semi-ellipsoidal load, evaluating curves in planes oriented in three directions, assuming Poisson's ration = 0.5, and taking the reference pressure as the maximum of the ellipsoidal distribution, which is related to the inflation pressure.

All curves are kept dimensionless by taking the characteristic dimension, B, as unity. By this technique, the curves are made valid for any load of the same proportions used here, regardless of size of contact area or magnitude of load. Depths and offsets measured in terms of one half the minor axis of the loaded area correspond to depths and offsets in the curves. Results are plotted for depths and offsets up to ten units. Deflections were calculated by integrating strains to a total depth of 500 units.

The completed curves appear in Figures 6 through 11. The orientation of the plane represented by each chart is shown on the figure, Figures 6 and 7 being for the transverse axis of load, Figures 8 and 9 for the diagonal and Figures 10 and 11 for the longitudinal axis. Vertical stresses are shown in Figures 6, 8 and 10. Stress in per cent of maximum contact pressure, P_{MAX}, is plotted to a logarithmic scale

on the horizontal axis, and depth to an arithmetic scale on the vertical axis. Each curve represents stress at a distance R from the center of the load along the line indicated on the figure. Numbers on the curves indicate the value of R in terms of B, one half the minor axis of the contact area.

Figures 7, 9 and 11 present deflection factors in a similar manner. Actual total deflection for a given point is determined by equation 8. To determine stress or deflection at some point under a multiple arrangement of wheels, the appropriate factors are read from the curves for each wheel, and added. The sum gives the effect of the load group. An example determination of stress and deflection under one wheel of a tandem axle is shown in Appendix B.

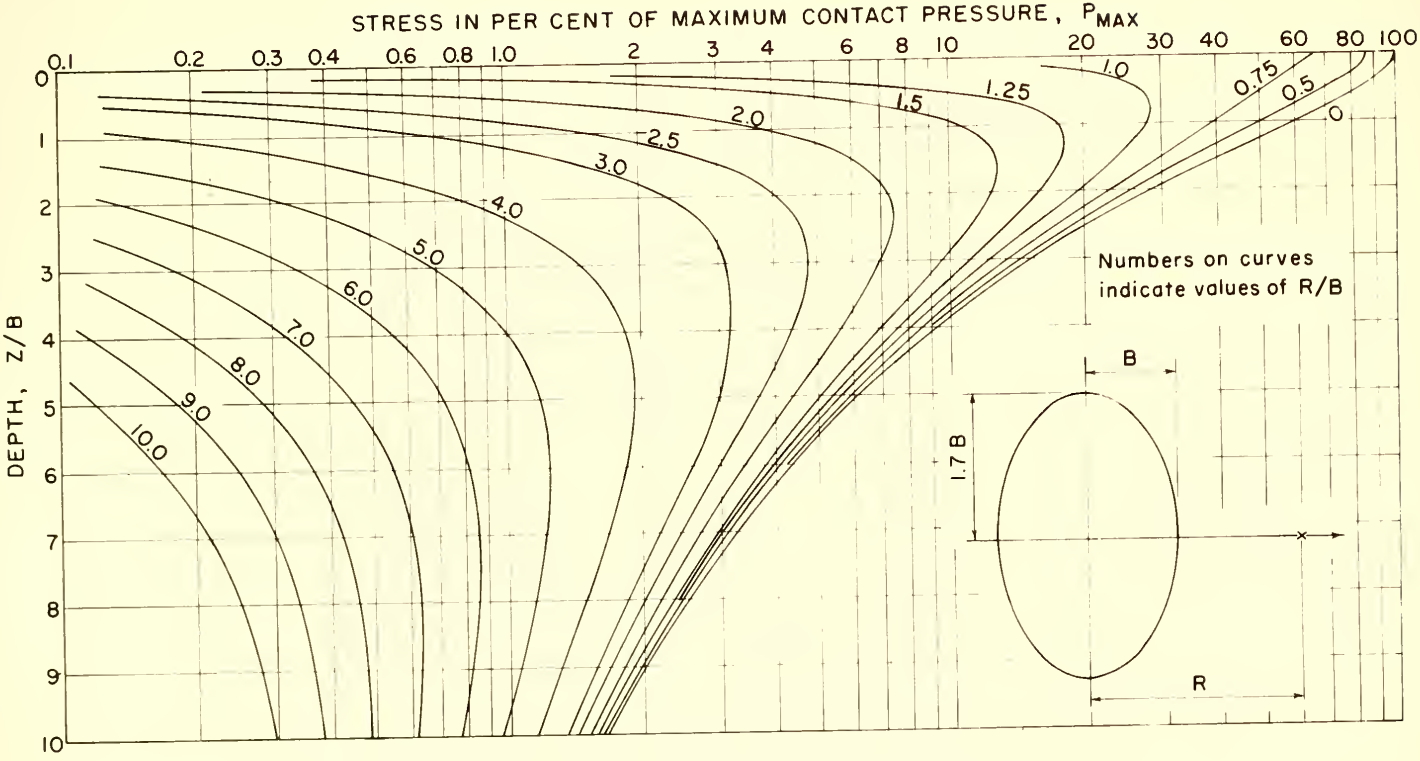

FIGURE 6. VERTICAL STRESS, σ_z. TRANSVERSE PLANE

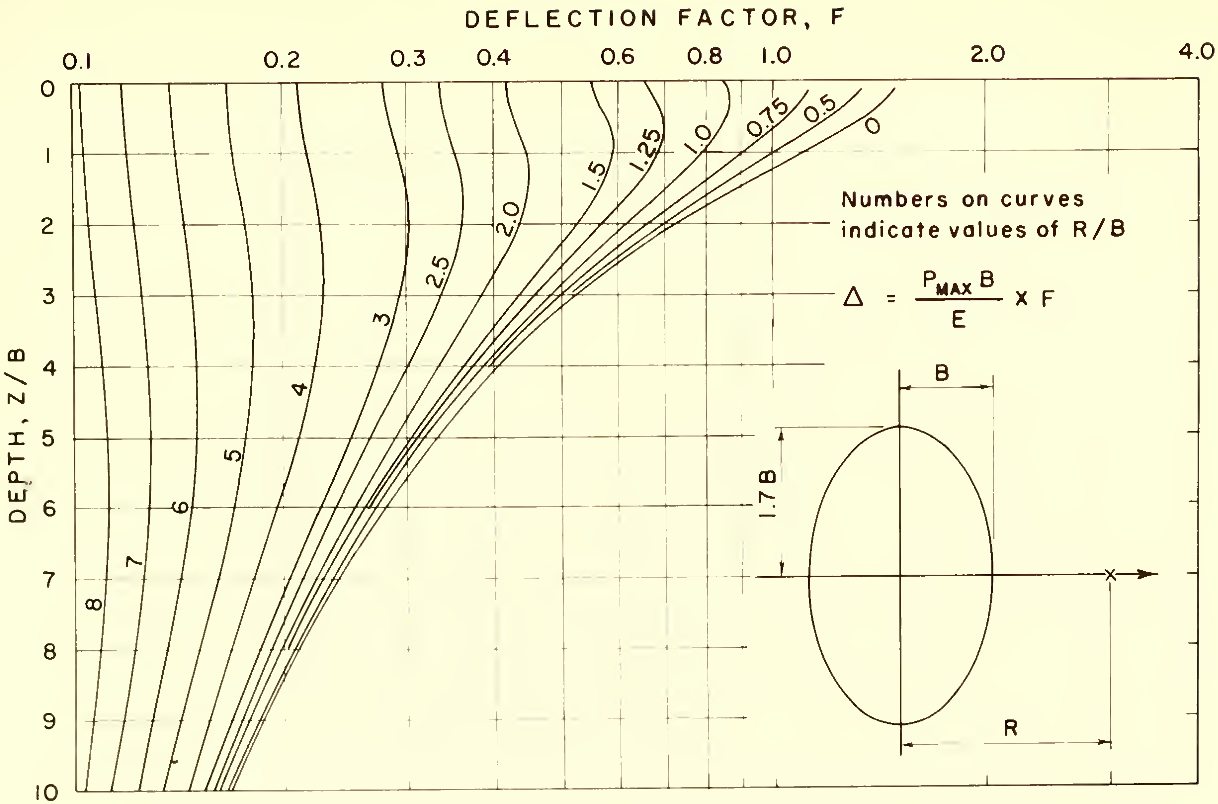

FIGURE 7. VERTICAL DEFLECTION, Δ. TRANSVERSE PLANE.
(POISSON'S RATIO = 0.5).

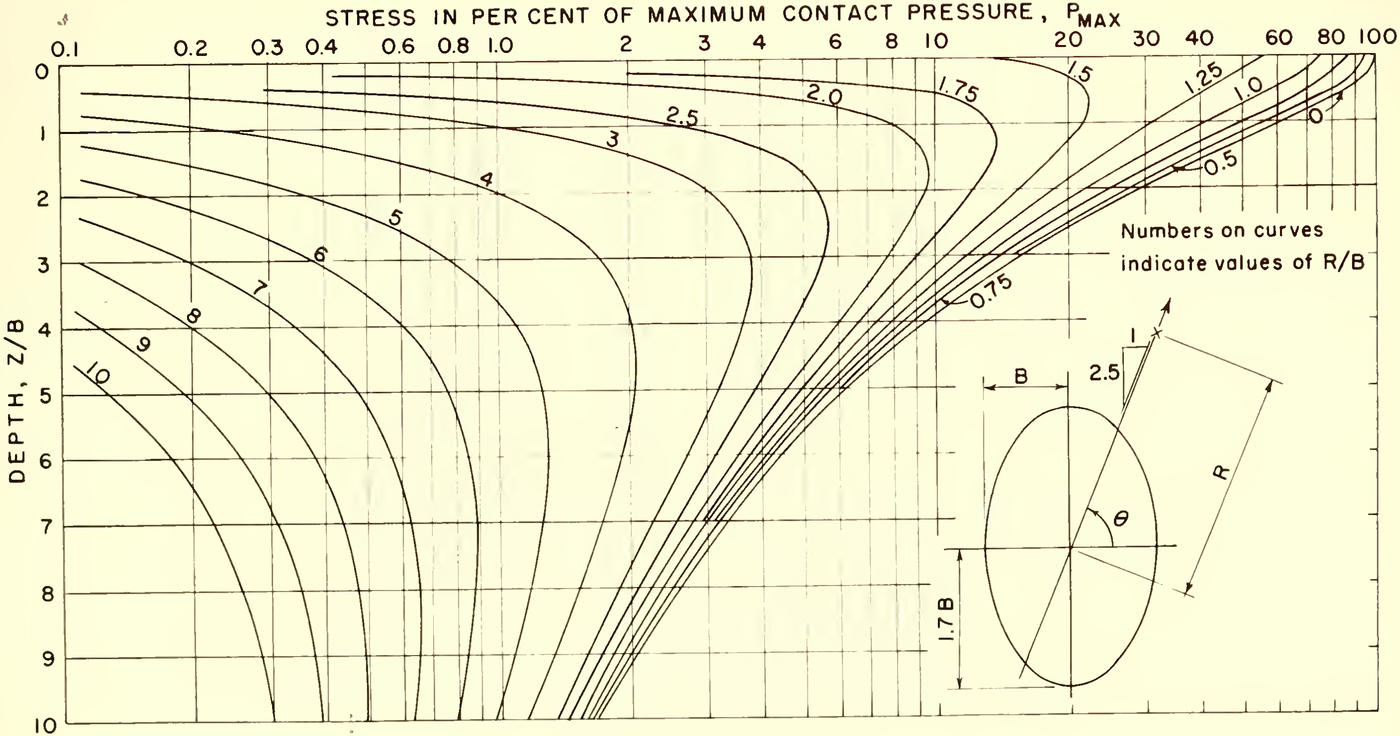

FIGURE 8. VERTICAL STRESS, σ_z. DIAGONAL PLANE ($\theta = 68.2°$).

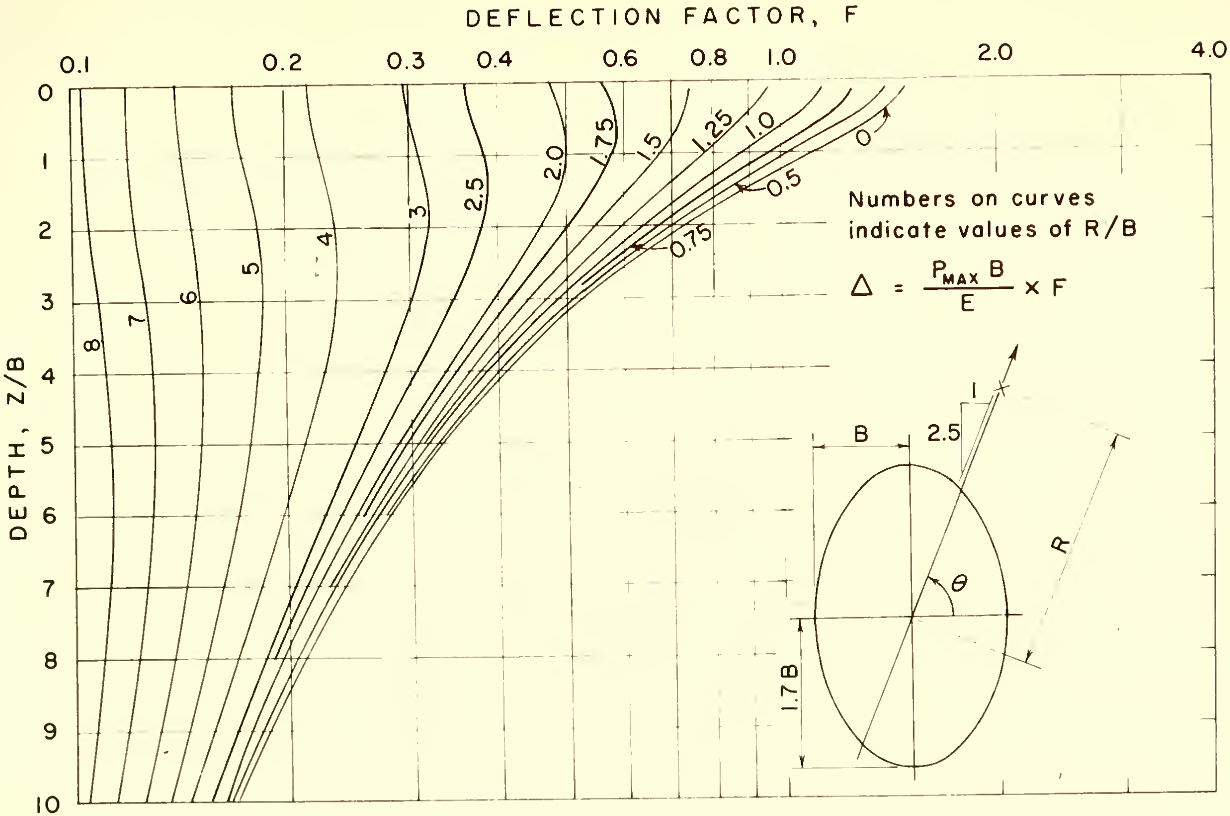

FIGURE 9. VERTICAL DEFLECTION, Δ. DIAGONAL PLANE ($\theta = 68.2°$). (POISSON'S RATIO = 0.5).

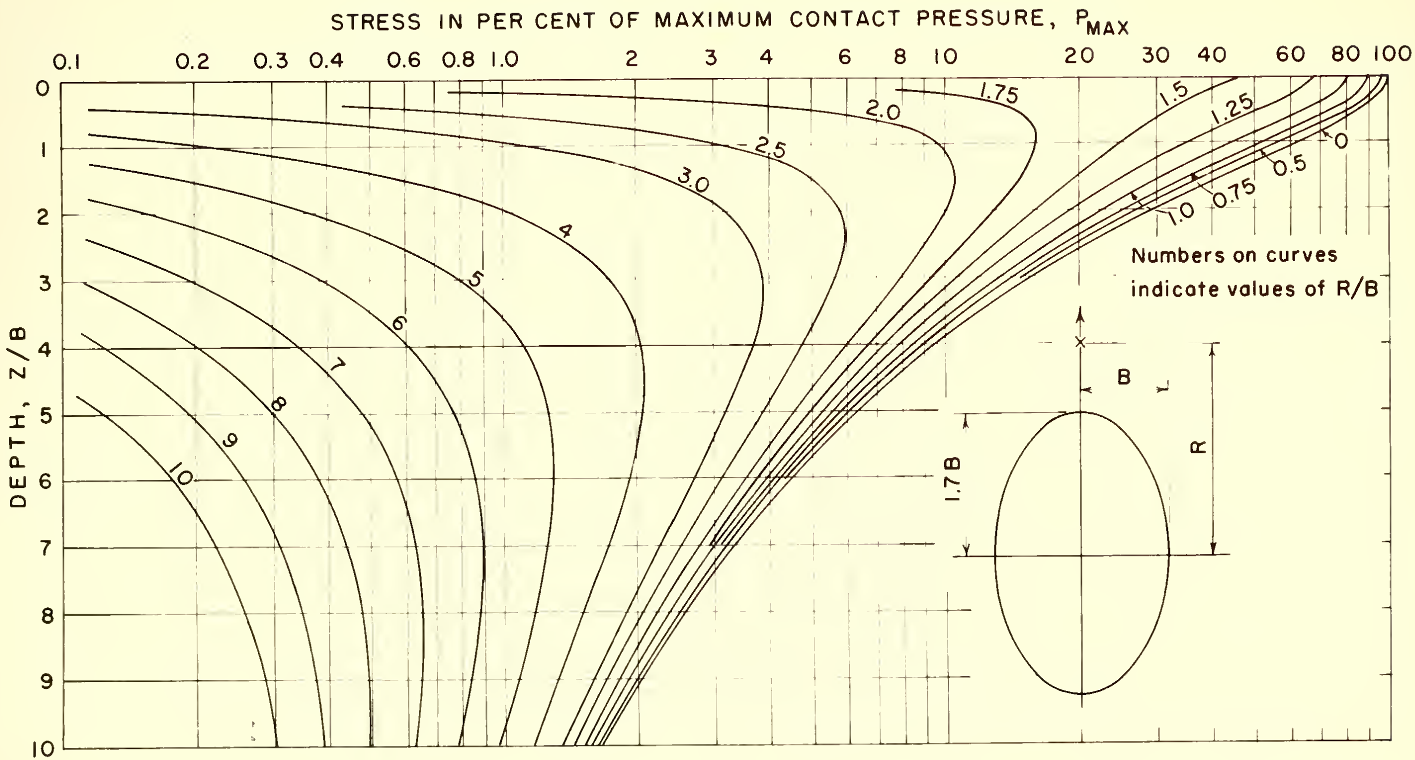

FIGURE 10. VERTICAL STRESS, σ_z. LONGITUDINAL PLANE.

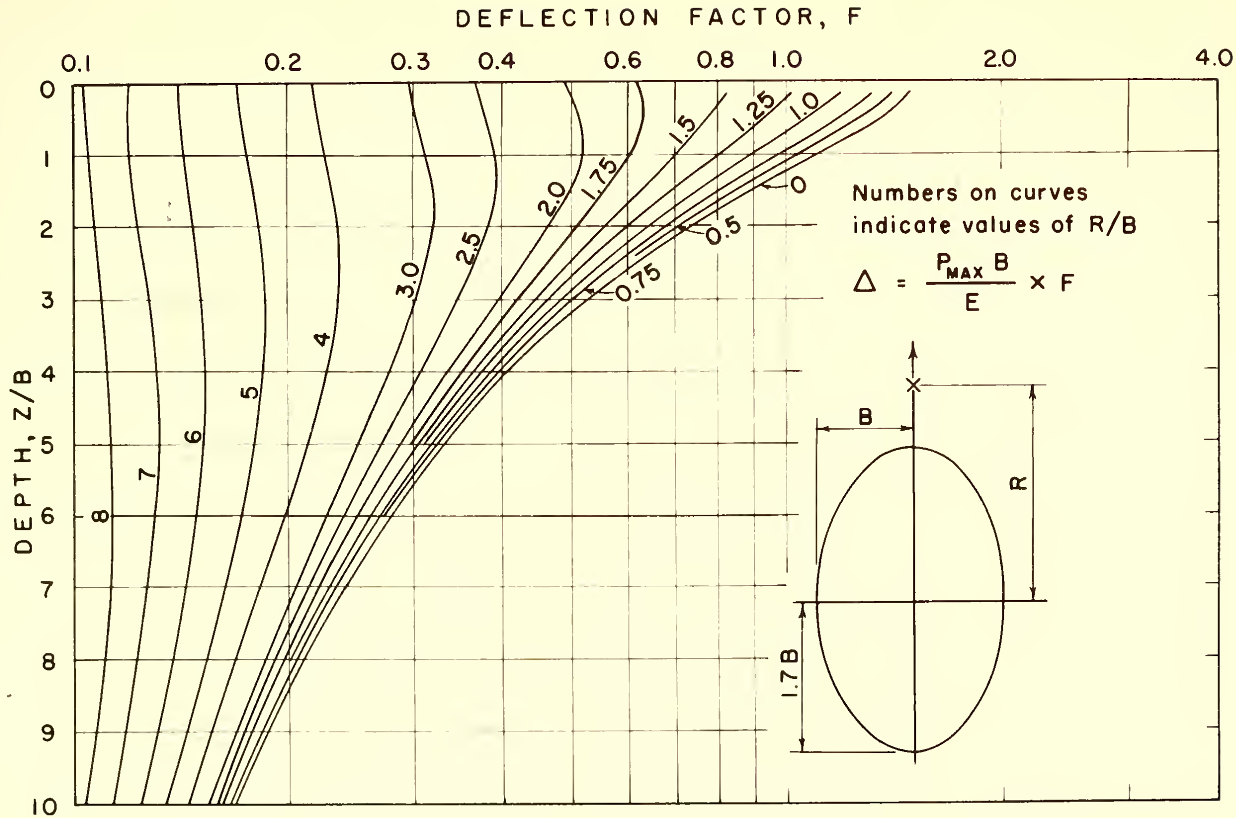

FIGURE 11. VERTICAL DEFLECTION, Δ. LONGITUDINAL PLANE. (POISSON'S RATIO = 0.5).

SUMMARY AND CONCLUSION

Summary

This study has produced a digital computer program to perform a numerical integration for determination of vertical stresses and deflections in pavements under wheel loads.

Loads were treated as semiellipsoidal distributions of pressure over the tire contact area. Boussinesq theory of stress distribution was applied, using Simpson's rule to integrate the effect of the entire loaded area. Elastic strains were calculated from the stresses, and integrated along vertical lines to determine total deflection factors.

Using the program reported here, curves of stress and deflection factor vs. depth were plotted for three different orientations of a vertical plane through the center of the load - transverse and longitudinal axes and one diagonal direction. The ratio of major to minor axis of load was taken as 1.7. The maximum pressure, at the center of the load, was used as a reference pressure. Poisson's ratio was taken as 0.5. Curves are provided for depths and offset distances up to five times the width of the load.

Recommendations

The following recommendations are made as a result of the study reported herein:

1. That further study be made of the distribution of pressure over the contact areas of tires -especially of truck tires.

2. That the computer program be used for further comparisons of different load distributions - in particular, the effect of uniform loads over elliptic contact areas compared to semi-ellipsoidal loads.

3. That a study of theoretical pavement deflections be made comparing results of three layer theory with those of this report.

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BIBLIOGRAPHY

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APPENDIX

APPENDIX A

Fortran Program for IBM 7090 Computer

```
      DIMENSION P2PI(25,25),R(25,25),U(25),V(25),W(25),U2(25),DY(25)
  110 FORMAT (F5.2,2I4,2F7.2,F5.2,I3)
  111 FORMAT (F5.2,5F7.2)
  112 FORMAT (1H ,F15.2,2F15.5)
  113 FORMAT (1H ,F15.2,3F15.5)
    3 READ INPUT TAPE 7,110,A,N,M,H,G,RATIO,IDENT
      IF(IDENT)2,4,4
    2 READ INPUT TAPE 7,111,DZ1,DZ2,DZ3,ZF1,ZF2,ZF3
    4 WRITE OUTPUT TAPE 6,110,A,N,M,H,G,RATIO
      DZ =DZ3
      PDELT = 0.0
      DELTA = 0.0
      SUMD = 0.0
      K = 1
      IZ = 1
      E = M+1
      F = N+1
      DX = 2.0*A/F
      X = DX-A
      DO 11 I=1,N
      YMAX = SQRTF(1.0-X*X/(A*A))
      DY(I) = 2.0*YMAX/E
      Y = DY(I)-YMAX
      DO 10 J=1,M
      P2PI(I,J) = (SQRTF(1.0-X*X/(A*A)-Y*Y))/(2.0*3.1416)
      R(I,J) = SQRTF((X-H)*(X-H)+(Y-G)*(Y-G))
   10 Y = Y+DY(I)
   11 X = X+DX
      M1 = M-1
      N1 = N-1
      Z = ZF3
    1 DO 16 I=1,N
      DO 15 J=1,M
      Q = SQRTF(Z*Z+R(I,J)*R(I,J))
      Q3 = Q*Q*Q
      Q5 = Q3*Q*Q
      S = R(I,J)*R(I,J)+Z*Z+Z*Q
      V(J) = 3.0*P2PI(I,J)*Z*Z*Z/Q5
      V2 = P2PI(I,J)*(3.0*R(I,J)*R(I,J)*Z/Q5-(1.0-2.0*RATIO)/S)
      V3 = P2PI(I,J)*(2.0*RATIO-1.0)*(Z/Q3-1.0/S)
      W(J) = V(J)-RATIO*(V2+V3)
   15 CONTINUE
```

```
      I2 = 1
 69   EVENS = 0.0
      DO 5 J=2,M1,2
  5   EVENS = EVENS+V(J)
      ODDS = 0.0
      DO 6 J=1,M,2
  6   ODDS = ODDS+V(J)
      SUM = DY(I)/3.0*(4.0*ODDS+2.0*EVENS)
      GO TO (72,73),I2
 72   U(I) = SUM
      I2 = 2
      DO 74 J=1,M
 74   V(J) = W(J)
      GO TO 69
 73   U2(I) = SUM
 16   CONTINUE
      I3 = 1
 79   EVENS = 0.0
      DO 7 I=2,N1,2
  7   EVENS = EVENS + U(I)
      ODDS = 0.0
      DO 8 I=1,N,2
  8   ODDS = ODDS + U(I)
      SUM = DX/3.0*(4.0*ODDS + 2.0*EVENS)
      GO TO (77,78),I3
 77   SIGZ = SUM
      I3 = 2
      DO 80 I=1,N
 80   U(I) = U2(I)
      GO TO 79
 78   DDELT = SUM
      GO TO (50,51),K
 50   SUMD = SUMD+DDELT
      DELTA = SUMD*DZ/3.0+PDELT
      SUMD = SUMD+DDELT
      K = 2
      WRITE OUTPUT TAPE 6,113,Z,SIGZ,DDELT,DELTA
      GO TO (90,91,92),IZ
 51   SUMD = SUMD+4.0*DDELT
      WRITE OUTPUT TAPE 6,112,Z,SIGZ,DDELT
      K = 1
      GO TO 41
 90   IF(Z-ZF2)100,100,41
100   DZ = DZ2
      PDELT = DELTA
      SUMD = DDELT
101   IZ = IZ+1
 41   Z = Z-DZ
      GO TO 1
 91   IF(Z-ZF1)102,102,41
102   DZ = DZ1
      PDELT = DELTA
```

```
      SUMD = DDELT
      GO TO 101
 92   IF(Z-DZ)3,3,41
      END
```

APPENDIX B

Example Determination of Stress and Deflection

As an example of the use of the curves presented in this report, the vertical stress and deflection at a depth of 36 in. beneath one wheel of a dual-tandem aircraft gear, at rated load, is determined. The gear configuration, as shown in Figure 12, consists of dual wheels spaced at 31-1/4 in. center to center and tandem axles at 61-1/4 in. center to center. Total gear load is 178,000 lb, inflation pressure is 176 psi, and contact area is 267 sq in. The modulus of elasticity of the subgrade is 2700psi.

$$\text{Contact area} = 1.7\pi B^2 = 267 \text{ sq in.}$$

$$B = \sqrt{\frac{267}{1.7\pi}} = 7.07 \text{ in.}$$

$$\frac{Z}{B} = \frac{36}{7.07} = 5.1$$

$$P_{MAX} = 1.32 \times 176 = 232 \text{ psi}$$

$$R_c = \sqrt{(31.25)^2 + (61.25)^2} = 68.7 \text{ in.}$$

Wheel	R	R/B	$z(\%P_{MAX})$	F
A	0.0	0.0	6.00 (Fig. 6)	0.33 (Fig. 7)
B	61.25	8.7	0.23 (Fig. 10)	$0.1^{\pm}$ (Fig. 11)
C	68.7	9.7	0.15 (Fig. 8)	$0.0^{\pm}$ (Fig. 9)
D	31.25	4.4	1.70 (Fig. 6)	0.19 (Fig. 7)
		Σ	8.08	0.62

$$\sigma_z = \frac{8.08}{100} \times 232 = 18.7 \text{ psi}$$

$$\Delta = \frac{232 \times 7.07}{2700} \times 0.62 = 0.38 \text{ in.}$$

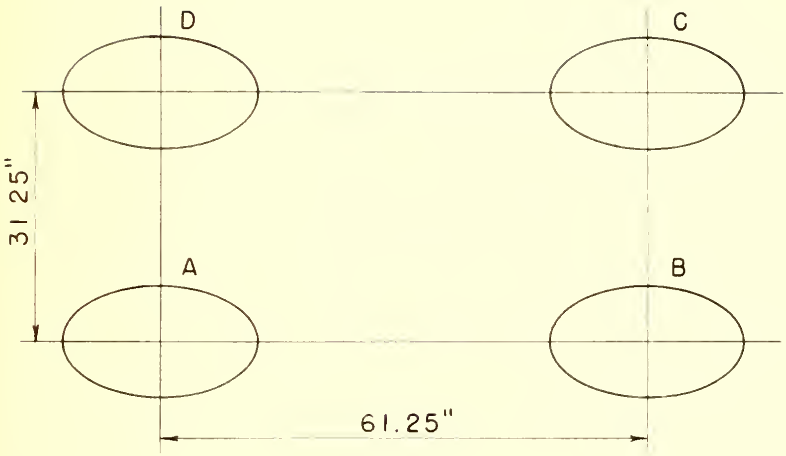

FIGURE 12. EXAMPLE GEAR CONFIGURATION.